Bounds on the Solutions of Difference Equations and Spline Interpolation at Knots

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ABSTRACT

We prove several comparison theorems for difference equations and discuss their application to spline interpolation at knots.

INTRODUCTION

Given a strictly increasing sequence

$$\Delta: \cdots < t_{-1} < t_0 < t_1 < \cdots,$$

$$\lim_{i\to-\infty} t_i = a, \qquad \lim_{i\to\infty} t_i = b, \qquad -\infty \leqslant a < b \leqslant \infty, \qquad I=(a,b),$$

we define

$$\mathcal{S}_n(\Delta) = \left\{ S : S \in C^{n-1}(I),\ S|_{(t_i,t_{i+1})} \in \pi_n,\ i \in Z = \{0, \pm 1, \pm 2, \dots\} \right\},$$

where $\pi_n = \{$polynomials of degree $\leqslant n\}$. $\mathcal{S}_n(\Delta)$ is the class of spline functions of degree n with knots on Δ.

PROBLEM. Find conditions on Δ which imply that for any bounded sequence $\dots, y_{-1}, y_0, y_1, \dots$ there exists a unique bounded $S \in \mathcal{S}_n(\Delta)$ such

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that

$$S(t_i)=y_i, \qquad i\in Z. \tag{0.1}$$

When a mesh Δ has this property, we will say that the problem (0.1) is *solvable*.

For $n=1$, (0.1) is solvable for any mesh. When $n=2$ the problem is also elementary. We will not go into the simple details of its analysis. For $n\geqslant 3$, the problem becomes difficult. The case $n=3$ was recently investigated by de Boor [1]. For arbitrary n and equally spaced partitions, $\Delta t_i=t_{i+1}-t_i=$ constant, the problem is studied in Schoenberg [8]; for a geometric mesh, $\Delta t_i/\Delta t_{i-1}=$constant, it is studied in [5].

In this paper we will study the general problem and assume from now on that $n\geqslant 3$. Our hope was to prove a conjecture made in [5]; however, our efforts were unsuccessful in this regard. A statement of this conjecture and our main results on (0.1) are discussed in Sec. 1.

The approach we use to study this interpolation problem replaces (0.1) by an (equivalent) first order difference equation on R^{n-1} with variable coefficients. This fact is our departure from (0.1) to the question of estimating the growth of solutions of a homogeneous difference equation

$$x^{i+1}=A_i x^i, \qquad i\in Z, \tag{0.2}$$

where each A_i is an $n\times n$ oscillation matrix and bounds on A_i are given. This information on (0.2) enables us to "solve" the inhomogeneous equation

$$x^{i+1}=A_i x^i+b^i, \qquad i\in Z, \tag{0.3}$$

in l^∞, by which we mean: given any bounded sequence $\{b^i : i\in Z\}$ in R^{n-1}, (0.3) has a unique bounded solution. When this is the case we will say (0.3) is solvable. Sections 2 and 3 of the paper give various conditions on A_i which guarantee that (0.3) is solvable.

1. SPLINE INTERPOLATION

We begin by reviewing some facts needed in the analysis of (0.1). To this end, we define

$$p_0(t)=1-t^n,$$

$$p_j(t)=\frac{t^j}{j!}(1-t^{n-j}), \qquad j=1,2,\ldots,n-1,$$

$$p_n(t)=t^n.$$

These polynomials have the following properties: for $i=0,1,\ldots,n-1$, $j=1,\ldots,n-1$,

$$p_0^{(i)}(0)=\delta_{0i}, \qquad p_0(1)=0,$$

$$p_j(0)=p_j(1)=0, \qquad p_j^{(i)}(0)=\delta_{ij},$$

$$p_n^{(i)}(0)=0, \qquad p_n(1)=1.$$

Hence any $S\in \mathbb{S}_n(\Delta)$ which satisfies (0.1) may be expressed, on $[t_i,t_{i+1}]$, as

$$S(t)=y_i p_0\left(\frac{t-t_i}{\Delta t_i}\right)+y_{i+1}p_n\left(\frac{t-t_i}{\Delta t_i}\right)+\sum_{j=1}^{n-1} S^{(j)}(t_i)(\Delta t_i)^j p_j\left(\frac{t-t_i}{\Delta t_i}\right),$$

$\Delta t_i=t_{i+1}-t_i$. Since $S\in C^{n-1}(I)$, we can differentiate this equation $n-1$ times, evaluate at $t=t_{i+1}$ and obtain the relations

$$(\Delta t_i)^l S^{(l)}(t_{i+1})=y_i p_0^{(l)}(1)+y_{i+1}p_n^{(l)}(1)+\sum_{j=1}^{n-1} S^{(j)}(t_i)(\Delta t_i)^j p_j^{(l)}(1),$$

$$l=1,\ldots,n-1. \tag{1.1}$$

We define vectors and matrices,

$$T_{lj}=-p_j^{(l)}(1), \qquad l,j=1,2,\ldots,n-1,$$

$$(x^i)_l=(-1)^i(\Delta t_{i-1})^l S^{(l)}(t_i), \qquad l=1,\ldots,n-1, \quad i\in Z,$$

$$(b^i)_l=(-1)^{i+1}\left[\,y_i p_0^{(l)}(1)+y_{i+1}p_n^{(l)}(1)\right], \qquad l=1,2,\ldots,n-1, \quad i\in Z,$$

$$D(m)=\operatorname{diag}\{m,m^2,\ldots,m^{n-1}\},$$

and

$$T(m)=TD(m).$$

Then (1.1) becomes

$$x^{i+1}=T(m_i)x^i+b^i, \qquad i\in Z, \quad x^i\in R^{n-1}, \tag{1.2}$$

where $m_i = \Delta t_i / \Delta t_{i-1}$. Clearly, there exists a positive constant d such that

$$\sup_{x \in I} |S(x)| \leqslant d \sup_{i \in Z} (\|x^i\|_\infty + |y_i| + |y_{i+1}|),$$

where $\|\cdot\|_\infty$ is the max norm on R^{n-1}. Thus the solvability of (1.2) implies the solvability of (0.1). We state below some facts about the matrix $T(m)$ which are proved in [5].

For every $m > 0$, the matrix $T(m)$ is an oscillation matrix (see Sec. 3 for the definition of this term). The eigenvalues of $T(m)$, which we denote by $\lambda_{n,1}(m), \dots, \lambda_{n,n-1}(m)$, are positive and simple:

$$\lambda_{n,1}(m) > \cdots > \lambda_{n,n-1}(m) > 0.$$

Each eigenvalue is a strictly increasing function of m, mapping $[0, \infty)$ onto itself, and for $m \neq 1$ the eigenvalues are the unique zeros of the equation

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{m^k + \lambda} = 0. \tag{1.3}$$

The matrix T has the further property that

$$T^{-1} = DTD, \qquad D = D(-1), \tag{1.4}$$

and consequently,

$$\lambda_{n,i}(m^{-1}) = \lambda_{n,n-i}^{-1}(m), \qquad i = 1, \dots, n-1, \tag{1.5}$$

$$\lambda_{n,1}(m) \cdots \lambda_{n,n-1}(m) = m \cdot m^2 \cdots m^{n-1}. \tag{1.6}$$

We introduce positive constants $m_{n,1}, \dots, m_{n,n-1}$ defined uniquely by the equations

$$\lambda_{n,i}(m_{n,i}) = 1, \qquad i = 1, 2, \dots, n-1.$$

According to (1.5)

$$0 < m_{n,1} < m_{n,2} < \cdots < m_{n,n-1},$$

and

$$m_{n,i} = m_{n,n-i}^{-1}.$$

Thus, for n odd, $m_{n,i} \neq 1$, while for n even, $m_{n,n/2} = 1$. Furthermore, these numbers are the unique positive zeros of the equation

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{m^k+1} = 0$$

(exclusive of the root $m=1$). In addition, we know that for each $m>0$ the $(n+1)$th set of eigenvalues $\lambda_{n+1,1}(m), \ldots, \lambda_{n+1,n}(m)$ interlace the nth set, that is,

$$\lambda_{n+1,1}(m) > \lambda_{n,1}(m) > \cdots > \lambda_{n+1,n-1}(m) > \lambda_{n,n-1}(m) > \lambda_{n+1,n}(m).$$

Hence, it is also the case that the corresponding m-value interlace:

$$m_{n+1,1} < m_{n,1} < \cdots < m_{n+1,n-1} < m_{n,n-1} < m_{n+1,n}.$$

To state our main results we require the following notion of *local mesh ratio*:

$$m(\Delta) = \lim_{N\to\infty} \sup_{\substack{|i-j|=1 \\ |i| \geq N}} \frac{\Delta t_i}{\Delta t_j}.$$

For odd degree spline functions, $n = 2k+1$, we conjecture that for any mesh Δ with

$$m(\Delta) < m_{2k+1,k+1} \tag{1.7}$$

the problem (0.1) is solvable. This bound on the local mesh ratio is the best possible, and of course, $m_{2k+1,k+1} > 1$.

We will prove the following theorem.

THEOREM 1.1. *The function* $R(m) = m^{k(2k+1)}/\lambda_{2k+1,k}(m)$ *is a strictly increasing function of* m *mapping* $[0,\infty)$ *onto itself. Let*

$$R(m_k^*) = 1;$$

then for any mesh Δ *with* $m(\Delta) < m_k^*$ *the problem* (0.1) *is solvable.*

Let us observe that when $n=3$, we have $k=1$ and

$$R(m) = \frac{m^3}{\lambda_{3,1}(m)} = \lambda_{3,2}(m).$$

Hence $m_1^* = m_{3,2} = (3+\sqrt{5})/2$, and in this case Theorem 1.1 was proved by de Boor [1].

The first few values of $m_{2k+1,k+1}$ are

$$m_{3,2} = \frac{3+\sqrt{5}}{2} = 2.61803,$$

$$m_{5,3} = 1.4163,$$

$$m_{7,4} = 1.1990.$$

There is a convenient lower bound for m_k^* which we will now describe. We introduce the Appell sequence $A_n(x;\lambda)$, $n=0,1,2,\dots,$ with generating function

$$\frac{e^{xz}}{e^z-\lambda} = \sum_{n=0}^{\infty} A_n(x;\lambda)z^n, \qquad \lambda \neq 1.$$

The equation

$$A_n(0;-\lambda)=0$$

has $n-1$ positive simple zeros,

$$\mu_{n,1} > \cdots > \mu_{n,n-1} > 0,$$

which are the eigenvalues of T, that is, $\lambda_{n,i}(1)=\mu_{n,i}$, $i=1,2,\dots,n-1$ [5]. Since

$$\begin{aligned}(m_k^*)^{k(2k+1)} &= \lambda_{2k+1,k}(m_k^*) > \lambda_{2k+1,k}(1)\\ &= \mu_{2k+1,k},\end{aligned}$$

the number $(\mu_{2k+1,k})^{[k(2k+1)]^{-1}}$ is a lower bound for m_k^*, and thus it may replace m_k^* in Theorem 1.1.

For a periodic mesh Δ we can give a complete solution to the problem (0.1).

Let us assume that

$$0=t_0<t_1<\cdots<t_{N-1}<t_N=1, \qquad N \geq 1,$$

and

$$t_{i+N}=t_i+1, \qquad i\in Z.$$

In this case we will say Δ is *N-periodic.*

Let us note that $A_n(x;-1)=E_n(x)$ is (up to a multiplicative constant) the nth Euler polynomial. The nth Bernoulli polynomial is defined by the generating function

$$\frac{z}{e^z-1}e^{xz}=\sum_{n=0}^{\infty}B_n(x)z^n.$$

The 1-periodic extension of the Bernoulli polynomial will be denoted by $\bar{B}_n$. Thus

$$\bar{B}_n(x)=B_n(x), \qquad 0\leqslant x\leqslant 1,$$

and for all x,

$$\bar{B}_{n+1}(x+1)=\bar{B}_n(x).$$

According to the defining equation for $B_n(x)$ above, it may be verified that

$$B_n(x+1)-B_n(x)=\frac{x^{n-1}}{(n-1)!}, \qquad n\geqslant 1.$$

Hence $B_n^{(i)}(1)=B_n^{(i)}(0)$, $i=0,1,\ldots,n-2$, and thus $\bar{B}_n$ is in $C^{n-2}(-\infty,\infty)$. Similarly, we define $\bar{A}_n(x;\lambda)$ by demanding that

$$\bar{A}_n(x;\lambda)=A_n(x;\lambda), \qquad 0\leqslant x\leqslant 1,$$

$$\bar{A}_n(x+1;\lambda)=\lambda\bar{A}_n(x;\lambda), \qquad -\infty<x<\infty.$$

Using the defining relation for $A_n(x;\lambda)$ we see that

$$A_n(x+1;\lambda)-\lambda A_n(x;\lambda)=\frac{x^n}{n!},$$

and consequently $\bar{A}_n\in C^{n-1}(-\infty,\infty)$. Finally we will make use of the notation $\partial_i a_{ij}=a_{i+1,j}-a_{i,j}$ for the (partial) difference of a_{ij} with respect to i.

THEOREM 1.2. *Let Δ be an N-periodic partition. When N is odd, the problem* (0.1) *is solvable if and only if*

$$\det_{i,j=0,1,\dots,N-1} \left\| \bar{E}_n(t_i - t_j) \right\| \neq 0.$$

When N is even, the problem (0.1) *is solvable if and only if*

$$\det_{i,j=0,\dots,N-1} \left\| \partial_j \partial_i \bar{B}_{n+1}(t_i - t_j) \right\| \neq 0.$$

Obviously for an N-periodic partition we have

$$T(m_{i+N}) = T(m_i), \qquad i \in Z.$$

In Sec. 2 we prove that any difference equation (0.3) which has an N-periodic coefficient matrix, $A_{i+N} = A_i$, $i \in Z$, is solvable if and only if the matrix $A_{N-1} \cdots A_0$ has no eigenvalues on the unit circle. The matrix $T_N = T(m_{N-1}) \cdots T(m_0)$ is an oscillation matrix and hence has $n-1$ positive simple eigenvalues. Thus (0.1) is solvable if and only if all the eigenvalues of T_N are different from 1. Let's now explore some alternative descriptions of the eigenvalues of T_N which will lead us to the easy proof of Theorem 1.2.

LEMMA 1.1. *Let Δ be an N-periodic partition. Then there exists a nonzero $S \in \mathcal{S}_n(\Delta)$ satisfying*

$$\begin{aligned} S(x+1) &= \lambda S(x), \\ S(t_i) &= 0, \qquad i = 0, 1, \dots, N-1, \end{aligned} \tag{1.8}$$

if and only if $(-1)^N \lambda$ is an eigenvalue of T_N.

Proof. This lemma is an immediate consequence of the difference equation (1.2). ■

LEMMA 1.2. *For $\lambda \neq 1$, $S \in \mathcal{S}_n(\Delta)$ satisfies the functional equation* (1.8) *if and only if S is expressible as a linear combination of $\bar{A}_n(x - t_0; \lambda), \dots, \bar{A}_n(x - t_{N-1}; \lambda)$:*

$$S(x) = \sum_{j=0}^{N-1} c_j \bar{A}_n(x - t_j; \lambda). \tag{1.9}$$

When $\lambda=1$, *then* S *satisfies* (1.8) *if and only if*

$$S(x)=\sum_{j=0}^{N} c_j\bar{B}_{n+1}(x-t_j)+c_{-1} \tag{1.10}$$

for some constants $c_0,\dots,c_N$ *with* $\sum_{j=0}^{N} c_j=0$.

Proof. When $\lambda\neq 1$, we choose $c_0,\dots,c_{N-1}$ so that $H(x)=S(x)-\sum_{j=0}^{N-1}c_j\bar{A}_n(x-t_j;\lambda)$ is in $C^n(-\infty,\infty)$. Hence H is a polynomial of degree $\leqslant n$ which satisfies the functional equation

$$H(x+1)=\lambda H(x), \qquad \lambda\neq 1.$$

Clearly, this fact implies $H\equiv 0$. When $\lambda=1$ the proof is similar; the right hand side of (1.10) is in $\mathbb{S}_n(\Delta)$ because $\sum_{j=0}^{N} c_j=0$. ■

The proof of Theorem 1.2 now easily follows from Lemma 1.1 and Lemma 1.2.

Proof. If N is odd, then (0.1) is solvable if and only if (1.8) has only the zero solution for $\lambda=-1$. According to Lemma 1.2 this happens exactly when

$$\det_{i,j=0,1,\dots,N-1}\left\|E_n(t_i-t_j)\right\|\neq 0.$$

The remainder of the proof follows similarly.

Since the eigenvalues of T_N are positive we have proved along the way that

$$\det_{i,j=0,1,\dots,N-1}\left\|\bar{A}_n(t_i-t_j;\lambda)\right\|\neq 0$$

if N is odd and $\lambda\geqslant 0$ or N is even and $\lambda\leqslant 0$. The sign of this determinant as $\lambda\to 0$ is easily seen to be $\operatorname{sgn}(-1/\lambda)^{N-1}(-1)^{nN}$, and thus we have

COROLLARY 1.1. *For any* $0=t_0<\cdots<t_{N-1}<1$,

$$(-1)^{nN}\det_{i,j=0,1,\dots,N-1}\left\|\bar{A}_n(t_i-t_j;\lambda)\right\|>0$$

if N *is odd with* $\lambda\geqslant 0$ *or if* N *is even with* $\lambda\leqslant 0$.

Returning to Lemma 1.1, let us point out that when n is odd, the only spline $S \in \mathcal{S}_n(\Delta)$ satisfying

$$S(x+1) = -S(x),$$

$$S(t_i) = 0, \qquad i = 0, 1, \dots, N-1,$$

is identically zero. For, if $n = 2m-1$, then integration by parts yields

$$\int_0^1 \left[S^{(m)}(x) \right]^2 dx = 0.$$

Hence S is a polynomial of degree $\leqslant m-1$, and now, because

$$S^{(i)}(1) = -S^{(i)}(0), \qquad i = 0, 1, \dots, m-1,$$

it easily follows that $S \equiv 0$.

Consequently we have

COROLLARY 1.2. *Let Δ be an N-periodic partition. If n is odd, then* (0.1) *is solvable.*

This corollary is a special case of a recent result of de Boor [2] on the solvability of (0.1) when n is odd and the *global* mesh ratio of Δ is bounded.

In the remainder of this section we will give an indication of the methods we use to prove Theorem 1.1. For this purpose, we require some further notation.

The eigenvalues of an $n \times n$ matrix A, arranged in decreasing order of magnitude, will be denoted by $\lambda_1(A), \dots, \lambda_n(A)$:

$$|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)| \geqslant 0.$$

We introduce an ordering $\prec$ on oscillation matrices as follows:

$$\underline{A} \prec \overline{A}$$

means that any minor of $\underline{A}$ is $\leqslant$ to the respective minor of $\overline{A}$, that is,

$$\det\left(\underline{A}_{i_p j_m}\right)_1^l = \underline{A}\begin{pmatrix} i_1, \dots, i_l \\ j_1, \dots, j_l \end{pmatrix} \leqslant \overline{A}\begin{pmatrix} i_1, \dots, i_l \\ j_1, \dots, j_l \end{pmatrix} = \det\left(\overline{A}_{i_p j_m}\right)_1^l$$

for all $1 \leqslant i_1 < \cdots < i_l \leqslant n, \qquad 1 \leqslant j_1 < \cdots < j_l \leqslant n.$

Let

$$D = \operatorname{diag}\{-1, +1, \dots, (-1)^{n-1}\},$$

$$\delta_1(\underline{A}, \bar{A}) = \lambda_1(\underline{A})$$

and

$$\delta_p(\underline{A}, \bar{A}) = \frac{\lambda_1(\underline{A}) \cdots \lambda_p(\underline{A})}{\lambda_1(\bar{A}) \cdots \lambda_{p-1}(\bar{A})}, \qquad p = 2, \dots, n.$$

The following result is a portion of our main result of Sec. 3.

THEOREM 1.3. *The difference equation*

$$x^{i+1} = A_i x^i + b^i, \qquad i \in Z,$$

is solvable if there exist oscillation matrices $\underline{A}, \bar{A}, \underline{B}, \bar{B}$ *such that (for* $|i|$ *sufficiently large)*

$$\begin{aligned} &\underline{A} \prec A_i \prec \bar{A}, && i \in Z,\\ &\underline{B} \prec D A_i^{-1} D \prec \bar{B}, && i \in Z, \end{aligned}$$

and an integer r, $1 \leqslant r \leqslant n$, *with* $\delta_p(\underline{A}, \bar{A}) > 1$, $p = 1, \dots, r$, *and* $\delta_p(\underline{B}, \bar{B}) > 1$, $p = 1 \dots n - r$.

Our proof of Theorem 1.1 is based on this result:

Proof of Theorem 1.1. Let $m(\Delta) \leqslant m < m_k^*$; then

$$\begin{aligned} &T(m^{-1}) \prec T(m_i) \prec T(m), && i \in Z,\\ &T(m^{-1}) \prec D T^{-1}(m_i) D \prec T(m), && i \in Z \end{aligned}$$

($|i|$ sufficiently large). Since $\lambda_{2k+1,p}(m^{-1}) < \lambda_{2k+1,p}(m) < \lambda_{2k+1,p-1}(m)$,

$$\begin{aligned} \delta_p &= \delta_p\left(T(m^{-1}), T(m)\right)\\ &= \frac{\lambda_{2k+1,1}(m^{-1}) \cdots \lambda_{2k+1,p}(m^{-1})}{\lambda_{2k+1,1}(m) \cdots \lambda_{2k+1,p-1}(m)} \end{aligned}$$

is a decreasing sequence of p. Thus according to Theorem 1.3 with $r=k$, the problem (0.1) is solvable provided that

$$\delta_k\left(T(m^{-1}),T(m)\right)=\frac{\lambda_{2k+1,1}(m^{-1})\cdots\lambda_{2k+1,k}(m^{-1})}{\lambda_{2k+1,1}(m)\cdots\lambda_{2k+1,k-1}(m)}>1.$$

Equivalently, from (1.5) and (1.6) we observe that

$$\frac{\lambda_{2k+1,1}(m)\cdots\lambda_{2k+1,k-1}(m)}{\lambda_{2k+1,1}(m^{-1})\cdots\lambda_{2k+1,k}(m^{-1})}$$

$$=\lambda_{2k+1,1}(m)\cdots\lambda_{2k+1,k-1}(m)\lambda_{2k+1,2k}(m)\cdots\lambda_{2k+1,k+1}(m)$$

$$=\frac{m\cdots m^{2k}}{\lambda_{2k+1,k}(m)}=R(m).$$

Since each $\lambda_{2k+1,i}(m)$ is strictly increasing, mapping $[0,\infty)$ onto itself, the above equation shows that $R(m)$ does likewise. Hence $R(m)<1$, because $m<m_k^*$ and $R(m_k^*)=1$. This completes the proof. ∎

Let us also observe that since

$$R(m)=\left[\lambda_{2k+1,k}(m^{-1})\right]^{-1}\frac{\lambda_{2k+1,1}(m)}{\lambda_{2k+1,1}(m^{-1})}\cdots\frac{\lambda_{2k+1,k-1}(m)}{\lambda_{2k+1,k-1}(m^{-1})},$$

it follows that

$$\lambda_{2k+1,k+1}(m)<R(m),\qquad m\in(0,\infty),\quad k>1.$$

Hence

$$m_k^*<m_{2k+1,k+1},$$

and our result is not the best possible for $k>1$.

2. SOME GENERAL REMARKS ON DIFFERENCE EQUATIONS

Let us begin by discussing some general aspects of the difference equation

$$x^{i+1}=A_ix^i+b^i,\qquad i\in Z. \tag{2.1}$$

The approach we follow in this section is based on some elementary properties of the spectrum of a bounded linear operator on a Banach space. This technique is not satisfactory in dealing with the problem on spline interpolation referred to in the introduction. However, the discussion in this section will prepare the way for our main results in Sec. 3.

In (2.1), $A_i=(a^i_{\mu\nu})_1^n$ is an $n\times n$ real matrix, and $x^i=(x_1^i,\dots,x_n^i)$ and $b^i=(b_1^i,\dots,b_n^i)$ are real vectors. Let $\|\cdot\|$ be a norm on R^n, and $\|x\|_\infty=\max\{|x_j|:1\leqslant j\leqslant n\}$ be the standard max norm of x. B will be the Banach space of all (ordered) vector sequences $\xi=\langle x^i:i\in Z\rangle$ with norm $\|\xi\|=\sup\{\|x^i\|:i\in Z\}$ induced by the vector norm $\|\cdot\|$ on R^n. Since all norms on R^n are equivalent, B may be identified with l^∞.

With a given sequence of $n\times n$ matrices $\{A_i:i\in Z\}$ we associate the operator $\mathcal{A}=\langle A_i\rangle$ defined by $(\mathcal{A}\xi)_i=A_{i-1}x^{i-1}$, $i\in Z$. When the sequence $\{A_i:i\in Z\}$ is uniformly bounded, $\mathcal{A}$ maps B into itself, and letting $\beta=\langle b^{i-1}:i\in Z\rangle$, Eq. (2.1) becomes

$$\xi=\mathcal{A}\xi+\beta. \tag{2.2}$$

Thus the statement that for any bounded sequence $\{b^i:i\in Z\}$ in R^n there exists a unique bounded sequence $\{x^i:i\in Z\}$ in R^n satisfying (2.1) is equivalent to $1\notin\sigma(\mathcal{A})=$spectrum of $\mathcal{A}$. When this is the case we will say (2.1) is solvable.

We will make use of the following lemma.

LEMMA 2.1. *Let $\mathcal{A},\mathcal{B}$ be bounded operators on B, and f a function analytic on $\sigma(\mathcal{A})$. If $f(\lambda)\notin\sigma(\mathcal{B})$ and $\|f(\mathcal{A})-\mathcal{B}\|<\|[f(\lambda)I-\mathcal{B}]^{-1}\|^{-1}$, then $\lambda\notin\sigma(\mathcal{A})$.*

Proof. First assume that $f(z)=z$ then for the operator $\mathcal{C}=(\lambda I-\mathcal{B})^{-1}(\lambda I-\mathcal{A})$ we have $\|\mathcal{C}-I\|<1$. Hence $\mathcal{C}$ has an inverse given by $\mathcal{C}^{-1}=\sum_{\nu=0}^{\infty}(I-\mathcal{C})^\nu$, and it follows that $\lambda\notin\sigma(\mathcal{A})$. Now, in the general case we have, by the above remarks, that $f(\lambda)\notin\sigma(f(\mathcal{A}))$. Hence by the spectral mapping theorem $\lambda\notin\sigma(\mathcal{A})$. ■

For real valued matrices A and B we say that $A\leqslant B$ if $a_{ij}\leqslant b_{ij}$, $i,j=1,\dots,n$ (the ordering $\leqslant$ for matrices should not be confused with the ordering $\prec$ on oscillation matrices introduced in Sec. 1), and A^+ will denote the matrix $(|a_{ij}|)_1^n$. We will denote the spectral radius of A by $\rho(A)$, and $x\leqslant y$, $x,y\in R^n$, means that $x_i\leqslant y_i$, $i=1,\dots,n$.

THEOREM 2.1. *Let A be a nonnegative irreducible matrix satisfying $\rho(A)<1$. Then either one of the following conditions implies that $1\notin\sigma(\mathcal{A})$:*

(i) *The sequence* $\{A_i : i \in Z\}$ *is uniformly bounded, and there exists an integer* $N \geqslant 0$ *such that*

$$(A_{i+N} \cdots A_i)^+ \leqslant A, \qquad i \in Z. \tag{2.3}$$

(ii) *The sequence* $\{A_i^{-1} : i \in Z\}$ *is uniformly bounded, and there exists an integer* $N \geqslant 0$ *such that*

$$\left(A_{i+N}^{-1} \cdots A_i^{-1}\right)^+ \leqslant A, \qquad i \in Z. \tag{2.4}$$

Proof. As A is nonnegative and irreducible, the Perron-Frobenius theorem tells us that there exists a positive vector $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i > 0$, $i = 1, \dots, n$, such that $A\alpha = \rho(A)\alpha$. Let $\|x\| = \max\{|x_j|/|\alpha_j| : 1 \leqslant j \leqslant n\}$; then (2.3) implies

$$\left| \frac{\Sigma_\nu (A_{i+N} \cdots A_i)_{\mu\nu} x_\nu}{\alpha_\mu} \right| \leqslant \frac{\Sigma_\nu A_{\mu\nu} |x_\nu|}{|\alpha_\mu|} \leqslant \rho(A)\|x\|,$$

and hence $\|A_{i+N} \cdots A_i\| \leqslant \rho(A)$. We have, for $\xi \in B$, $(\mathcal{C}^{N+1}\xi)_i = A_{i-1} \cdots A_{i-N-1} x^{i-N-1}$, $i \in Z$. Thus $\|\mathcal{C}^{N+1}\| \leqslant \rho(A) < 1$, and using Lemma 2.1 with $\mathcal{B} = 0$, $f(z) = z^{N+1}$, we conclude that $1 \notin \sigma(\mathcal{C})$. Similarly (ii) implies $1 \notin \sigma(\mathcal{C})$. ■

As a corollary to Theorem 2.1 we have

COROLLARY 2.1. *The problem* (0.1) *is solvable provided that there exists an* m_0 *such that either* $m_i \leqslant m_0 < m_{n,1}$, $i \in Z$, *or* $m_i \geqslant m_0 > m_{n,n-1}$, $i \in Z$.

Proof. If $m_i \leqslant m_0 < m_{n,1}$, then $T(m_i) \leqslant T(m_0)$ and $\rho(T(m_0)) = \lambda_{n,1}(m_0) < 1$. If $m_i \geqslant m_0 > m_{n,n-1}$, then $\left(T^{-1}(m_i)\right)^+ \leqslant \left(T^{-1}(m_0)\right)^+$ and $\rho\left(\left(T^{-1}(m_0)\right)^+\right) = \left(\lambda_{n,n-1}(m_0)\right)^{-1} < 1$. Thus in either case the corollary follows from Theorem 2.1. ■

When we specialize Corollary 2.1 to $n = 3$ and combine it with Theorem 1.1, we see that (0.1) is solvable provided that the sequence $\{m_i : i \in Z\}$ lies strictly within one of the intervals

$$\left(0, \frac{3-\sqrt{5}}{2}\right), \quad \left(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right), \quad \left(\frac{3+\sqrt{5}}{2}, \infty\right).$$

We now examine the case that (2.1) is a constant coefficient difference equation.

Recall that $\lambda_1(A),\ldots,\lambda_n(A)$ are the eigenvalues of the $n\times n$ matrix A arranged in decreasing order of magnitude,

$$|\lambda_1(A)|\geqslant|\lambda_2(A)|\geqslant\cdots\geqslant|\lambda_n(A)|.$$

THEOREM 2.2. *Given any bounded sequence $\{b_i : i\in Z\}$ in R^n, there exists a unique bounded solution to the constant coefficient difference equation*

$$x^{i+1}=Ax^i+b^i,\qquad i\in Z, \tag{2.5}$$

if and only if $|\lambda_j(A)|\neq 1$, $1\leqslant j\leqslant n$.

Proof. For $n=1$, (2.5) reduces to a scalar difference equation

$$x_{i+1}=\lambda x_i+b_i,\qquad i\in Z. \tag{2.6}$$

For $|\lambda|\neq 1$, the unique bounded solution to (2.6) is given by

$$x_i=\begin{cases}\displaystyle\sum_{j=1}^{\infty}\lambda^{j-1}b_{i-j}, & |\lambda|<1,\\ \displaystyle-\sum_{-\infty}^{0}\lambda^{j-1}b_{i-j}, & |\lambda|>1.\end{cases} \tag{2.7}$$

Now, suppose $n>1$ and $|\lambda_j(A)|\neq 1$, $j=1,\ldots,n$. Let $A=V^{-1}JV$ be the Jordan normal form for A. If we set $z^i=Vx^i$, $c^i=Vb^i$, then (2.5) becomes

$$z^{i+1}=Jz^i+c^i,\qquad i\in Z. \tag{2.8}$$

If J is composed of k Jordan blocks, then (2.8) separates into k independent difference equations. Thus we may assume without loss of generality that

$$J=\begin{bmatrix}\lambda & 1 & & & & \\ & \lambda & 1 & & \mathbf{0} & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & \mathbf{0} & & & \ddots & 1\\ & & & & & \lambda\end{bmatrix}.$$

The last component of the vector z^i is determined by a scalar difference equation (2.6) with $|\lambda| \neq 1$ and thus can be solved for uniquely. Proceeding by backward substitution, all the components of z^i are uniquely determined.

Conversely, suppose that $|\lambda_j(A)| = |\lambda| = 1$ for some j, $1 \leqslant j \leqslant n$, and let $x, y \in C^n$, $Ax = \lambda x$, $yA = \lambda y$, be a corresponding right and left eigenvector of λ respectively. The sequence $x^i = \lambda^i x$, $i \in Z$, is a bounded solution to the difference equation $x^{i+1} = Ax^i$. Thus for $b^i = 0$, $i \in Z$, (2.5) has more than one solution. Furthermore, if we let $b^i = \lambda^{i+1} y$, then $\{b^i : i \in Z\}$ is a bounded sequence in C^n, and an easy computation shows that $\lim_{i\to\infty} |(x^i, y)| = \infty$ for any solution to (2.5). Thus there exists no bounded solution to (2.5) for this choice of b^i. ■

From this lemma we deduce that for the operator $\mathcal{C} = \langle A \rangle$,

$$\sigma(I - \mathcal{C}) = \sum_{j=1}^{n} \{\lambda : |\lambda - 1| = |\lambda_j(A)|\}. \tag{2.9}$$

Since the spectrum of $(I - \mathcal{C})^{-1}$ is the image of the set (2.9) under the transformation $f(z) = z^{-1}$, the spectral radius of $(I - \mathcal{C})^{-1}$ is given by

$$\rho\big((I - \mathcal{C})^{-1}\big) = \max\Big\{ \big|1 - |\lambda_k(A)|\big|^{-1} : 1 \leqslant k \leqslant n \Big\}.$$

For any norm on B, the spectral radius of $I - \mathcal{C}$ is related to its norm by the inequality

$$\rho\big((I - \mathcal{C})^{-1}\big) \leqslant \big\|(I - \mathcal{C})^{-1}\big\|. \tag{2.10}$$

When A is similar to a diagonal matrix, it is simple to assign a norm on B which gives equality in (2.10).

LEMMA 2.2. *Assume that A is similar to a diagonal matrix Λ ($A = V^{-1}\Lambda V$), and $|\lambda_j(A)| \neq 1$, $j = 1, 2, \ldots, n$. Let $\|x\| = \|Vx\|_\infty$; then $\|(I - \mathcal{C})^{-1}\| = \max\Big\{ \big|1 - |\lambda_k(A)|\big|^{-1} : 1 \leqslant k \leqslant n \Big\}$.*

Proof. The proof of this lemma follows directly from (2.7). ■

THEOREM 2.3. *Let A be a matrix which is similar to a diagonal matrix, let $|\lambda_j(A)| \neq 1$, $j = 1, \ldots, n$, and assume that $\|x\| = \|Vx\|_\infty$, where $A = V^{-1}\Lambda V$. Then either one of the following conditions implies that* (2.2) *is solvable*:

(i) *The sequence $\{A_i : i \in Z\}$ is uniformly bounded, and there exist an* *integer* $N \geqslant 0$ *and a* ρ, $0 < \rho < 1$, *such that*

$$\|A_{i+N} \cdots A_i - A^{N+1}\| \leqslant \rho \min\left\{ \left| 1 - |\lambda_k(A)|^{N+1} \right| : 1 \leqslant k \leqslant n \right\}.$$

(ii) *The sequence* $\{A_i^{-1} : i \in Z\}$ *is uniformly bounded,* $\lambda_j(A) \neq 0$, $j = 1, \ldots, n$, *and there exist an integer* $N \geqslant 0$ *and a* ρ, $0 < \rho < 1$, *such that*

$$\|A_i^{-1} \cdots A_{i+N}^{-1} - A^{-N-1}\| \leqslant \rho \min\left\{ \left| 1 - |\lambda_k(A)|^{-N-1} \right| : 1 \leqslant k \leqslant n \right\}.$$

Proof. The proof of this theorem follows from Lemma 2.2 and Lemma 2.1 with $f(z) = z^{N+1}$, $N \in Z$, $\mathfrak{B} = \langle \mathfrak{A}^{N+1} \rangle$ and $\lambda = 1$. ■

Specializing Theorem 2.3 (i) to $N = 0$ and applying it to (1.2), it is easy to prove

COROLLARY 2.2. *Let* $m \notin \{m_{n,1}, \ldots, m_{n,n-1}\}$. *Then there exists an* $\varepsilon = \varepsilon(m) > 0$ *such that if* $|m_i - m| \leqslant \varepsilon$, $i \in Z$, *then* (0.1) *is solvable.*

Finally, let us observe that Theorem 2.2 extends to difference equations with periodic coefficients.

THEOREM 2.4. *Assume that there exists an integer* $N \geqslant 1$ *such that* $A_{i+N} = A_i$, $i \in Z$. *Then the system* (2.1) *is solvable if and only if* $|\lambda_j(A_{N-1} \cdots A_0)| \neq 1$, $j = 1, \ldots, n$.

Proof. It suffices to show that $1 \notin \sigma(\mathfrak{A}^N)$ if and only if $|\lambda_j(A_{N-1} \cdots A_0)| \neq 1$, $j = 1, \ldots, n$. The difference equation $\xi = \mathfrak{A}^N \xi + \beta$ separates into N constant coefficient difference equations given by

$$x^{(i+1)N+j} = A_{j-1} \cdots A_0 A_{N-1} \cdots A_j x^{iN+j} + b^{(i+1)N+j-1},$$

$$i \in Z, \quad j = 0, \ldots, N-1.$$

Therefore this theorem is a consequence of Theorem 2.2. ■

3. OSCILLATION MATRICES

We will now examine the difference equation (2.1) when each A_i is an oscillation matrix.

The matrix A is called totally positive (strictly totally positive) provided that all the minors of A are nonnegative (positive), that is,

$$A\begin{pmatrix} i_1,\dots,i_k \\ j_1,\dots,j_k \end{pmatrix} = \det(a_{i_\mu j_\nu})_1^k \geqslant 0,$$

$$1 \leqslant i_1 < \cdots < i_k \leqslant n, \quad 1 \leqslant j_1 < \cdots < j_k \leqslant n, \quad k=1,\dots,n.$$

A is an oscillation matrix if A is totally positive and some power of A is strictly totally positive.

Let $x=(x_1,\dots,x_n)$ be a vector in R^n. We denote by $S^+(x)$ $[S^-(x)]$ the maximum [minimum] number of sign changes in x when the zero components of x are replaced by either 1 or -1.

The following useful theorem is due to Gantmacher and Krein (cf. [3]).

THEOREM 3.1. *Let A be an oscillation matrix. Then A has n distinct positive eigenvalues, $\lambda_1(A)>\lambda_2(A)>\cdots>\lambda_n(A)>0$. Furthermore, if v^i is the eigenvector corresponding to the* ith *eigenvalue $\lambda_i(A)$, then*

$$p-1 \leqslant S^-\left(\sum_{i=p}^{q} \alpha_i v^i\right) \leqslant S^+\left(\sum_{i=p}^{q} \alpha_i v^i\right) \leqslant q-1.$$

The pth compound of A, denoted by $C_p(A)$, is the $\binom{n}{p}\times\binom{n}{p}$ matrix composed of all $p\times p$ minors of A arranged in lexicographic order. We introduced in Sec. 1 an ordering on $n\times n$ matrices by defining $A \prec B$ provided that $C_p(A)\leqslant C_p(B)$, $p=1,2,\dots,n$, where $\leqslant$ is the ordering on matrices defined in Sec. 2 preceding Theorem 2.1.

We will also use the following notation, also introduced in Sec. 1:

$$\delta_1(\underline{A},\overline{A}) = \lambda_1(\underline{A}), \tag{3.1}$$

$$\delta_p(\underline{A},\overline{A}) = \frac{\lambda_1(\underline{A})\cdots\lambda_p(\underline{A})}{\lambda_1(\overline{A})\cdots\lambda_{p-1}(\overline{A})}, \qquad p=2,\dots,n. \tag{3.2}$$

The main theorem of this section is

THEOREM 3.2. *Let $\underline{A},\overline{A},\underline{B},\overline{B}$ be $n\times n$ oscillation matrices. Each of the following conditions implies that* (2.1) *is solvable*:

(a) $\underline{A}\prec A_i \prec \overline{A}$, $i\in Z$, *and there exists an integer k, $1\leqslant k\leqslant n$, such that $\delta_p(\underline{A},\overline{A})>1$, $p=1,\dots,k$, and $\delta_p(\overline{A},\underline{A})<1$, $p=k+1,\dots,n$.*

(b) $\underline{A} \prec A_i \prec \bar{A}$, $\underline{B} \prec DA_i^{-1}D \prec \bar{B}$, *and there is a* k, $1 \leqslant k \leqslant n$, *such that* $\delta_p(\underline{A},\bar{A}) > 1$, $p = 1,\dots,k$, *and* $\delta_p(\underline{B},\bar{B}) > 1$, $p = 1,\dots,n-k$.

In the case that $\underline{A} = \bar{A}$, (a) becomes $\underline{A} = A_i = \bar{A}$, $i \in Z$. Hence (2.1) is a constant coefficient difference equation, and (a) means that $\underline{A}$ has k eigenvalues greater than 1 and $n-k$ less than 1. Thus in this case the theorem follows from Theorem 2.2.

The proof of Theorem 3.2 will be developed through a series of ancillary results which are of some independent interest. To this end, we require some further notation.

Let u^i, $i = 1,\dots,n$, be n vectors in R^n. Then the wedge product of $u^1,\dots,u^p$, $1 \leqslant p \leqslant n$, denoted by $v = u^1 \wedge u^2 \wedge \cdots \wedge u^p$, is a vector in

$$\bigwedge_{j=1}^{p} R^n = R^{\binom{n}{p}}$$

whose coordinates are given by

$$v(j_1,\dots,j_p) = \det\left(u_{j_\mu}^{\nu}\right)_{\nu,\mu=1}^{p}$$

$$= \begin{vmatrix} u_{j_1}^1 & \cdots & u_{j_1}^p \\ \vdots & & \vdots \\ u_{j_p}^1 & \cdots & u_{j_p}^p \end{vmatrix},$$

where the indices $(j_1,\dots,j_p)$, $1 \leqslant j_1 < \cdots < j_p \leqslant n$, are arranged in lexicographic order. We will call vectors $v^1,\dots,v^n$ a Chebyshev system if $v^1 \wedge \cdots \wedge v^j > 0$, $j = 1,\dots,n$. Chebyshev systems have the property that

$$S^+\left(\sum_{j=1}^{p} a_j v_j\right) \leqslant p-1, \quad p = 1,\dots,n; \qquad \sum_{j=1}^{p} a_j^2 \neq 0. \tag{3.3}$$

Also, an important part of the Gantmacher-Krein theorem is the fact that the eigenvectors of an oscillation matrix may be chosen so that $v^1 \wedge \cdots \wedge v^j > 0$, $j = 1,\dots,n$; see [3], Vol. II, p. 108.

The pth compound of A is the matrix induced by A on $\bigwedge_{j=1}^{p} R^n$, that is, $C_p(A) = \bigwedge_{j=1}^{p} A$. Hence

$$C_p(A)v = \bigwedge_{j=1}^{p} A\left(\bigwedge_{i=1}^{p} u^i\right) = Au^1 \wedge Au^2 \wedge \cdots \wedge Au^p,$$

and by the Cauchy-Binet formula,

$$\bigwedge_{i=1}^{p} AB=\left(\bigwedge_{i=1}^{p} A\right)\bigwedge_{i=1}^{p} B.$$

Thus the requirement that A should be totally positive means that $\bigwedge_{i=1}^{p}A$ is a nonnegative matrix for all p, $1\leqslant p\leqslant n$.

LEMMA 3.1. *Let $0\neq v\in R^n$ and $S^-(v)=p-1$, $1\leqslant p\leqslant n$. Then there exist linearly independent vectors $v^1,\dots,v^n$ with $v^1\wedge\cdots\wedge v^i\geqslant 0$ and $v^p=\pm v$.*

Proof. Suppose first that all the components of v are distinct from zero. Let $F_\sigma=(e^{-\sigma(i-j)^2})_1^n$; then it is well known that for any $\sigma>0$, F_σ is strictly totally positive (cf. [3]). Clearly, $F_\sigma\to I$ as $\sigma\to\infty$. Since $v=(v_1,\dots,v_n)$ has no zero components, for σ large we have $(F_\sigma v)_i v_i>0$, $i=1,\dots,n$. Let $\bar{D}=\operatorname{diag}\{d_1,\dots,d_n\}$, $d_i=v_i/(F_\sigma v)_i$, $i=1,\dots,n$. Then v is an eigenvector of the matrix $\bar{D}F_\sigma$, $\bar{D}F_\sigma v=v$. Since $\bar{D}F_\sigma$ is strictly totally positive, the condition $S^-(v)=p-1$ implies by Theorem 3.1 that $\lambda_p(\bar{D}F_\sigma)=1$. Furthermore, $\bar{D}F_\sigma$ has a corresponding eigensystem $\{v^i: 1\leqslant i\leqslant n\}$ which may be chosen so that $v=\pm v^p$. Thus the lemma is proved when v has no zero components.

Suppose now that some components of v vanish. Let $1\leqslant j_1<\cdots<j_r\leqslant n$ be the nonvanishing components of v $(r\geqslant p)$. Let $\bar{x}=(x_{j_1},\dots,x_{j_r})$. Now according to the above argument we can construct vectors $\bar{v}^1\wedge\cdots\wedge\bar{v}^i>0$, with $\bar{v}=\pm\bar{v}^p$. Extend $\bar{v}^i$, $i=1,\dots,n$, to R^n by defining its kth component, $k\neq j_1,\dots,j_r$, to be zero. Then it is easily seen that the resulting vectors $v^1,\dots,v^n$ satisfy the demand of the lemma. This ends the proof. ∎

THEOREM 3.3. *Let $\underline{A},\bar{A},A_i$, $i=0,1,2,\dots$, be oscillation matrices such that $\underline{A}\prec A_i\prec\bar{A}$, $i=0,1,2,\dots$. Let $x\neq 0$ and $S^-(x)=p-1$ for some p, $1\leqslant p\leqslant n$, and suppose $x^{i+1}=A_i x^i$, $i=0,1,2,\dots$, $x^0=x$. Then there exist constants $\underline{c},\bar{c}$ (depending on $x,\bar{A},\underline{A}$) such that*

$$\|x^i\|_\infty\geqslant\underline{c}\left[\delta_p(\underline{A},\bar{A})\right]^i,\qquad i=0,1,2,\dots,\tag{3.4}$$

and if $S^+(x^i)=p-1$, $i=0,1,2,\dots$, then

$$\|x^i\|_\infty\leqslant\bar{c}\left[\delta_p(\bar{A},\underline{A})\right]^i,\qquad i=0,1,2,\dots.\tag{3.5}$$

Before we begin the proof of this result, let us note that the hypothesis

$\underline{A} \prec \bar{A}$ implies that $\rho(C_p(\underline{A})) = \underline{\lambda}_1 \cdots \underline{\lambda}_p \leqslant \bar{\lambda}_1 \cdots \bar{\lambda}_p = \rho(C_p(\bar{A}))$, $p = 1, \ldots, n$. This fact is a consequence of the min-max characterization of the largest eigenvalue for a nonnegative matrix [7], applied to the pth compound of $\underline{A}$ and $\bar{A}$.

Proof. There exist, according to Lemma 3.1, linearly independent vectors $v^1, \ldots, v^n$, such that $v^1 \wedge \cdots \wedge v^j \geqslant 0$, $j = 1, \ldots, n$, and $v^p = \pm x$. Define $v^{r,i+1} = A_i v^{r,i}$, $i = 0, 1, \ldots$, and $v^{r,0} = v^r$, $r = 1, \ldots, n$. Thus the condition $\underline{A} \prec A_i \prec \bar{A}$, $i = 0, 1, \ldots$, implies

$$\left(\bigwedge_{j=1}^{r} \underline{A} \right)^i v^1 \wedge \cdots \wedge v^r \leqslant v^{1,i} \wedge \cdots \wedge v^{r,i} \leqslant \left(\bigwedge_{j=1}^{r} \bar{A} \right)^i v^1 \wedge \cdots \wedge v^r, \qquad r = 1, \ldots, n. \tag{3.6}$$

Since

$$p \| x^i \|_\infty \geqslant \frac{\| v^{1,i} \wedge \cdots \wedge v^{p,i} \|_\infty}{\| v^{1,i} \wedge \cdots \wedge v^{p-1,i} \|_\infty}, \tag{3.7}$$

[a fact easily proved by expanding $(v^{1,i} \wedge \cdots \wedge v^{p,i})$ $(j_1, \ldots, j_p)$ by its last column], we obtain

$$p \| x^i \| \geqslant \frac{\left\| \left(\bigwedge_{j=1}^{r} \underline{A} \right)^i v^1 \wedge \cdots \wedge v^p \right\|_\infty}{\left\| \left(\bigwedge_{j=1}^{r-1} \bar{A} \right)^i v^1 \wedge \cdots \wedge v^{p-1} \right\|_\infty}. \tag{3.8}$$

Recall that for any nonnegative primitive matrix A, we have $A^i \sim [\rho(A)]^i \xi \eta^T$, $i \to \infty$, where $\rho(A)$ is the spectral radius of A, and ξ, η are the corresponding (strictly positive) right and left eigenvectors of $\rho(A)$, respectively. Hence,

$$\left(\bigwedge_{j=1}^{r} \underline{A} \right)^i \sim (\underline{\lambda}_1 \ldots \underline{\lambda}_r)^i \underline{Q}_r, \qquad i = 0, 1, 2, \ldots, \qquad r = 1, 2, \ldots, n, \tag{3.9}$$

$$\left(\bigwedge_{j=1}^{r} \bar{A} \right)^i \sim (\bar{\lambda}_1 \ldots \bar{\lambda}_r)^i \bar{Q}_r, \qquad i = 0, 1, 2, \ldots, \qquad r = 1, 2, \ldots, n, \tag{3.10}$$

where $\underline{Q}_r$ and $\bar{Q}_r$ are fixed positive matrices depending only on $\underline{A}$ and $\bar{A}$.

Combining (3.9), (3.10) with (3.8) we obtain (3.4).

To prove (3.5) we use the additional hypothesis $S^+(x^i)=p-1$ and show that

$$\|x^i\|_\infty \leqslant \frac{\|v^{1,i}\wedge\cdots\wedge v^{p,i}\|_\infty}{\min\limits_{1\leqslant j_1<\cdots<j_{p-1}\leqslant n}(v^{1,i}\wedge\cdots\wedge v^{p-1,i})(j_1,\dots,j_{p-1})}. \tag{3.11}$$

For some k, $1\leqslant k\leqslant n$, we have $|x_k^i|=\|x^i\|_\infty$. Since $S^+(x^i)=p-1$, there exist indices $1\leqslant j_1<\cdots<j_p\leqslant n$ such that $x_{j_l}^i x_{j_{l+1}}^i\leqslant 0$, $l=1,\dots,p-1$, and $k\in\{j_1,\dots,j_p\}$. Now expand the determinant $(v^{1,i}\wedge\cdots\wedge v^{p,i})(j_1,\dots,j_p)$ by its last column:

$$\begin{aligned}&|(v^{1,i}\wedge\cdots\wedge v^{p,i})(j_1,\dots,j_p)|\\ &=\sum_{s=1}^{p}|v_{j_s}^{p,i}|(v^{1,i}\wedge\cdots\wedge v^{p-1,i})\,(j_1,\dots,j_{s-1},j_{s+1},\dots,j_p).\end{aligned} \tag{3.12}$$

Hence (3.11) easily follows. Substituting (3.6), (3.9), (3.10) into (3.11) gives (3.5). This completes the proof. ■

The hypothesis $\underline{A}\prec A_i\prec\overline{A}$, $i=0,1,\dots,$ in Theorem 3.3 means that

$$\bigwedge_{j=1}^{r}\underline{A}\leqslant\bigwedge_{j=1}^{r}A_i\leqslant\bigwedge_{j=1}^{r}\overline{A},\qquad r=1,\dots,n.$$

We may relax this hypothesis to the requirement that

$$\underline{\gamma}_i\bigwedge_{j=1}^{r}\underline{A}\leqslant\bigwedge_{j=1}^{r}A_i\leqslant\overline{\gamma}_i\bigwedge_{j=1}^{r}\overline{A},\qquad i=0,1,2,\dots,\qquad r=1,\dots,n, \tag{3.13}$$

and still obtain estimates for the norm $\|x^i\|_\infty$.

The usefulness of this observation is due to the fact that $\underline{A}\prec A_i\prec\overline{A}$, $i=0,1,2,\dots$ implies

$$\frac{\det\underline{A}}{\det A_i}\bigwedge_{j=1}^{r}D\underline{A}^{-1}D\leqslant\bigwedge_{j=1}^{r}DA_i^{-1}D\leqslant\frac{\det\overline{A}}{\det A_i}\bigwedge_{j=1}^{r}D\overline{A}^{-1}D,\ i=0,1,2,\dots, \tag{3.14}$$

where $D=\operatorname{diag}\{-1,\dots,(-1)^n\}$. This property of the ordering $\prec$ is a simple consequence of the formula for the minors of the inverse of a matrix [4].

The following theorem is an easily proven variation of Theorem 3.3.

THEOREM 3.4. *Let $\underline{A}$, $\bar{A}$, A_i, $i=0,1,\ldots$, be oscillation matrices satisfying* (3.13) *for some constants $\underline{\gamma}_i, \bar{\gamma}_i > 0$. Let $x \neq 0$, $S^-(x)=p-1$ for some p, $1 \leqslant p \leqslant n$, and suppose $x^{i+1}=A_i x^i$, $i=0,1,2,\ldots$, with $x^0=x$. Then there exist constants $\underline{c},\bar{c}$ (depending on $x,\bar{A},\underline{A}$) such that*

$$\|x^i\|_\infty \geqslant \underline{c}\frac{\underline{\gamma}_0\cdots\underline{\gamma}_{i-1}}{\bar{\gamma}_0\cdots\bar{\gamma}_{i-1}}\left[\delta_p(\underline{A},\bar{A})\right]^i, \qquad i=0,1,\ldots, \tag{3.15}$$

and if $S^+(x^i)=p-1$, $i=0,1,2,\ldots$, then

$$\|x^i\|_\infty \leqslant \bar{c}\frac{\bar{\gamma}_0\cdots\bar{\gamma}_{i-1}}{\underline{\gamma}_0\cdots\underline{\gamma}_{i-1}}\left[\delta_p(\bar{A},\underline{A})\right]^i, \qquad i=0,1,\ldots. \tag{3.16}$$

As a corollary to Theorem 3.4 we have the uniqueness assertion for the difference equation (2.1).

THEOREM 3.5. *Under the hypothesis of Theorem 3.2 the only bounded solution of the homogeneous difference equation $x^{i+1}=A_i x^i$, $i\in Z$, is the zero solution.*

Proof. We begin by showing that (a) implies uniqueness.

It is sufficient to prove that $x^0=0$. Let $S^-(x^0)=p-1$; then according to Theorem 3.3 and (a) it follows that $p \geqslant k+1$. Thus $S^-(Dx^0)=n-1-S^+(x^0) \leqslant n-1-S^-(x^0) \leqslant n-k-1$. Let $y^i=Dx^{-i}$; then $y_i=\underline{B}_i y^{i-1}$, where $B_i=DA_{-i}^{-1}D$. According to (3.14) the condition $\underline{A} \prec A_i \prec \bar{A}$, $i=0,\pm1,\pm2,\ldots$, implies

$$\frac{\det\underline{A}}{\det A_i}\bigwedge_{j=1}^{r} D\underline{A}^{-1}D \leqslant \bigwedge_{j=1}^{r} DA_i^{-1}D \leqslant \frac{\det\bar{A}}{\det A_i}\bigwedge_{j=1}^{r} D\bar{A}^{-1}D, \qquad i=0,\pm1,\pm2,\ldots.$$

Hence by Theorem 3.4, if $S^-(Dx_0)=q-1$, then there is a constant $\underline{c}>0$ such that

$$\|x^{-i}\|_\infty = \|y^i\|_\infty \geqslant \underline{c}\left\{\frac{\det\underline{A}}{\det\bar{A}}\,\frac{\lambda_n^{-1}(\underline{A})\cdots\lambda_{n-q+1}^{-1}(\underline{A})}{\lambda_n^{-1}(\bar{A})\cdots\lambda_{n-q+2}^{-1}(\bar{A})}\right\}^i, \qquad i=0,1,2,\ldots,$$

$$=\underline{c}\left\{\frac{\lambda_1(\underline{A})\cdots\lambda_{n-q}(\underline{A})}{\lambda_1(\bar{A})\cdots\lambda_{n-q+1}(\bar{A})}\right\}^i = \underline{c}\left[\delta_{n-q+1}(\bar{A},\underline{A})\right]^{-i}.$$

But $n-q+1 \geqslant n-(n-k)+1=k+1$; hence (a) implies $\lim_{i\to\infty}\|x^{-i}\|_\infty=\infty$. Thus $x^0=0$.

The proof of (b) proceeds in an analogous fashion. From (a) we conclude that $S^-(x^0)\geqslant k$, while if in the proof of part (a) we apply Theorem 3.3 directly to the matrices $B_i=DA_{-i}^{-1}D$, we obtain

$$\|x^{-i}\| \geqslant \underline{c}\left[\delta_q(\underline{B},\bar{B})\right]^i.$$

Since $S^-(x^0)\geqslant k$, we have $q-1=S^-(Dx^0)\leqslant n-k-1$, and hence $\lim_{i\to\infty}\|x^{-i}\|=\infty$. The theorem is therefore proved. ■

Now, let us turn to the question of the existence of a bounded solution to (2.1). We will show that for any bounded data $\{b^i : i\in Z\}$ in R^n, (2.1) has a unique bounded solution under the hypothesis of Theorem 3.2. We begin by recording below some familiar properties of totally positive matrices, some of which we have already used.

LEMMA 3.2. *Let A be an $n\times n$ matrix.*

(a) *If A is strictly totally positive, then $S^+(Ax)\leqslant S^-(x)$.*

(b) *If A has rank n and is totally positive, then $S^-(Ax)\leqslant S^-(x)$.*

(c) *If A is an oscillation matrix, then A^{n-1} is strictly totally positive.*

(d) *If $v^1,\dots,v^n$ is a Chebyshev system and A is an oscillation matrix, then $Av^1,\dots,Av^n$ is a Chebyshev system.*

(e) *If $v^1,\dots,v^n$ is a Chebyshev system, then $S^+(\Sigma_{k=1}^j\alpha_k v^k)\leqslant j-1$ for $\{\alpha_k\}$ such that $\Sigma_{k=1}^j\alpha_k^2\neq 0$, $j\leqslant n$.*

Proof. All the above properties are familiar facts from total positivity. For instance, (b) follows from the fact that A can be approximated by strictly totally positive matrices, while (d) follows from the Cauchy-Binet formula and the fact that the diagonal entries of $\wedge_{j=1}^r A$ are positive (see [4]). ■

LEMMA 3.3. *Let $\underline{A},\bar{A},A_i$, $i=0,1,2,\dots$, be $n\times n$ oscillation matrices such that $\underline{A}\prec A_i\prec\bar{A}$, $i=0,1,2,\dots$. Then there exist a constant $w>0$ (depending only on $\underline{A}$ and $\bar{A}$) and linear independent vectors $u^{i,1},\dots,u^{i,n}$, $i=0,1,2,\dots$, such that $S^+(u^{i,j})=S^-(u^{i,j})=j-1$, $j=1,\dots,n$; $u^{i+1,j}=A_iu^{i,j}$, $i=0,1,2,\dots$, $j=1,\dots,n$; $\|u^{0,j}\|_\infty=1$, $j=1,\dots,n$; and $u^{0,1}\wedge\cdots\wedge u^{0,n}\geqslant w$.*

Proof. Let $v^1,\dots,v^n$ be any Chebyshev system. Define $v^{i+1,j}=A_iv^{i,j}$, $i=0,1,\dots$, where $v^{0,j}=v^j$, $j=1,\dots,n$, and choose constants $\alpha_k^{i,j}$ such that $(\Sigma_{k=1}^j\alpha_k^{i,j}v^{i,k})_p=(-1)^p$, $p=1,\dots,j$. Thus $S^-(\Sigma_{k=1}^j\alpha_k^{i,j}v^{i,k})\geqslant j-1$. From

Lemma 3.2 we have

$$
\begin{aligned}
j-1 \leqslant S^{-}\left(\sum_{k=1}^{j} \alpha_k^{i,j} v^{i,k}\right) &= S^{-}\left(A_i \sum_{k=1}^{j} \alpha_k^{i,j} v^{i-1,k}\right) \\
&\leqslant S^{-}\left(\sum_{k=1}^{j} \alpha_k^{i,j} v^{i-1,k}\right) \\
&\leqslant \cdots \leqslant S^{-}\left(\sum_{k=1}^{j} \alpha_k^{i,j} v^{k}\right) \leqslant j-1.
\end{aligned}
$$

Also, because of Lemma 3.2,

$$
S^{+}\left(\sum_{k=1}^{j} \alpha_k^{i,j} v^{\rho,k}\right) \leqslant j-1 \qquad \text{for all} \quad \rho \geqslant 0.
$$

Hence

$$
S^{+}\left(\sum_{k=1}^{j} \alpha_k^{i,j} v^{\rho,k}\right) = S^{-}\left(\sum_{k=1}^{j} \alpha_k^{i,j} v^{\rho,k}\right) = j-1, \qquad 0 \leqslant \rho \leqslant i, \quad j=1,\ldots,n. \tag{3.17}
$$

We renormalize the constants $\alpha_k^{i,j}$ so that $\sum_{k=1}^{j}(\alpha_k^{i,j})^2=1$, $i=0,1,\ldots$, $j=1,\ldots,n$, and select a convergent subsequence $\lim_{i'\to\infty}\alpha_k^{i',j}=\alpha_k^j$. Define $u^{i,j}=\sum_{k=1}^{j}\alpha_k^j v^{i,k}$; then by (3.17)

$$
S^{-}(u^{i,j}) \leqslant j-1 \leqslant S^{+}(u^{i,j}).
$$

On the other hand, $S^{+}(u^{i,j}) \leqslant j-1$, since by Lemma 3.2, $v^{i,1},\ldots,v^{i,n}$ is a Chebyshev system. Hence $S^{+}(u^{i,j})=j-1$. Moreover, since $\underline{A}^{n-1} \prec A_{i+n-2}\cdots A_i$, we conclude that $A_{i+n-2}\cdots A_i$ is strictly totally positive. Thus

$$
\begin{aligned}
j-1 \leqslant S^{+}(u^{i+n-1,j}) &= S^{+}(A_{i+n-2}\cdots A_i u^{i,j}) \\
&\leqslant S^{-}(u^{i,j}) \leqslant j-1.
\end{aligned}
$$

Thus $S^{+}(u^{i,j})=S^{-}(u^{i,j})=j-1$. Next we show that $\{u^{i,j}\}$, $j=1,\ldots,n$, are linearly independent. However, if $\alpha_j^j = 0$, then $S^+(u^{0,j}) < j-1$ because $v^1,\dots,v^n$ is a Chebyshev system. But we know that $S^+(u^{0,j}) = j-1$. This is a contradiction. Let us now renormalize the vectors so that $\|u^{0,j}\|_\infty = 1$, $j = 1,\dots,n$. The proof will be complete provided we can show that there exists a universal positive lower bound for $u^{0,1} \wedge \cdots \wedge u^{0,n}$ which depends only on $\underline{A}$ and $\bar{A}$. Suppose to the contrary that a positive lower bound does not exist. Hence there is a sequence $\{A_i(r)\}$ such that $u^{0,1}(r) \wedge \cdots \wedge u^{0,m}(r)$ converges to zero as $r \to \infty$. We select a convergent subsequence $A_i(r') \to \hat{A}_i$, $u^{0,j}(r') \to \hat{u}^{0,j} = \sum_{k=1}^{j} \hat{\alpha}_k^j v^k$. As before, we show that $S^+(\hat{u}^{i,j}) = S^-(\hat{u}^{i,j}) = j-1$, $j = 1,\dots,n$, and $\hat{u}^{0,1},\dots,\hat{u}^{0,n}$ are linearly independent.

Thus $\hat{u}^{0,1} \wedge \cdots \wedge \hat{u}^{0,n} > 0$, so as not to contradict the fact that $S^+(\hat{u}^{0,j}) = j-1$, $j = 1,\dots,n$. But this is impossible, since $\hat{u}^{0,1} \wedge \cdots \wedge \hat{u}^{0,n} = 0$ by construction; thus we conclude that $w > 0$ exists, and the lemma is proved. ■

In our next lemma we prepare the way for the construction of Green's function for (2.1).

LEMMA 3.4. *Let the hypothesis of Theorem 3.2 hold. Then there are constants $c > 0$, $\rho < 1$ (depending only on $\underline{A}$, $\bar{A}$ and/or $\underline{B}, \bar{B}$) such that for every $b \in R^n$ there is a (unique) sequence $\{x^k : k \in Z\} \subseteq R^n$ with*

$$A_i x^i - x^{i+1} = 0, \qquad i \in Z - \{0\},$$

$$A_0 x^0 - x^1 = b,$$

and

$$\|x^i\|_\infty \leqslant c\rho^{|i|} \|b\|_\infty, \qquad i \in Z. \tag{3.18}$$

Proof. We begin by assuming that (a) is valid. Let $u_+^{i,1},\dots,u_+^{i,n}$ be the vectors constructed in Lemma 3.3—that is, $u_+^{i+1,j} = A_i u_+^{i,j}$, $i = 0,1,\dots$, $j = 1,\dots,n$, and

$$S^+(u_+^{i,j}) = S^-(u_+^{i,j}) = j-1, \qquad j = 1,\dots,n, \quad i = 0,1,\dots$$

—and extend them for $i = -1,-2,\dots$ so that

$$u_+^{i+1,j} = A_i u_+^{i,j}, \qquad i \in Z, \quad j = 1,\dots,n.$$

Since $S^-(u_+^{i,j}) = S^+(u_+^{i,j}) = j-1$, $i = 0,1,2,\dots$, we may appeal to (3.11) and conclude that

$$\|u_+^{i,j}\|_\infty \leqslant \frac{\|u_+^{i,1}\wedge\cdots\wedge u_+^{i,j}\|_\infty}{\min\limits_{1<k_1<\cdots<k_{j-1}\leqslant n}\left(u_+^{i,1}\wedge\cdots\wedge u_+^{i,j-1}\right)(k_1,\cdots,k_{j-1})}.$$

Also, from (3.9) and (3.10), there exist constants $\underline{c},\bar{c}$ depending only on $\underline{A}$ and $\bar{A}$ such that

$$\underline{c}(\underline{\lambda}_1\cdots\underline{\lambda}_r)^i\underline{Q}_r\leqslant\left(\bigwedge_{j=1}^{r}A\right)^i\leqslant\left(\bigwedge_{j=1}^{r}\bar{A}\right)^i\leqslant\bar{c}\left(\bar{\lambda}_1\cdots\bar{\lambda}_r\right)^i\bar{Q}_r,\quad i\geqslant n-1.$$

Thus we conclude that there is a constant $c>0$ such that

$$\|u^{i,j}\|_\infty\leqslant c\frac{\|u_+^{0,1}\wedge\cdots\wedge u_+^{0,j}\|_\infty}{\|u_+^{0,1}\wedge\cdots\wedge u_+^{0,j-1}\|_\infty}\left[\delta_j(\bar{A},\underline{A})\right]^i$$

$$\leqslant jc\|u_+^{0,j}\|_\infty\left[\delta_j(\bar{A},\underline{A})\right]^i,$$

$i=0,1,2,\ldots$. Thus the vectors $u_+^{i,k+1},\ldots,u_+^{i,n}$ are linearly independent solutions to the homogeneous difference equation

$$x^{i+1}=A_ix^i,\qquad i\in Z,$$

which decay exponentially fast as $i\to\infty$:

$$\|u_+^{i,j}\|_\infty\leqslant c_+\rho_+^i,\qquad i=0,1,2,\ldots,\quad j=k+1,\ldots,n,\quad \rho<1.$$

Similarly, for the "backward" equation determined by the matrix $B_i=DA_{-i}^{-1}D$, we construct by Lemma 3.3 linearly independent vectors $v^{i,1},\ldots,v^{i,n}$ such that

$$v^{i+1,j}=B_iv^{i,j},\qquad j=1,2,\ldots,n,\quad i\in Z,$$

$$S^+(v^{i,j})=S^-(v^{i,j})=j-1,\qquad i=0,1,\ldots,\quad j=1,2,\ldots,n.$$

Since

$$\frac{\det\underline{A}}{\det A_i}\bigwedge_{j=1}^{r}D\underline{A}^{-1}D\leqslant\bigwedge_{j=1}^{r}DA_i^{-1}D\leqslant\frac{\det\bar{A}}{\det A_i}\bigwedge_{j=1}^{r}D\bar{A}^{-1}D,$$

we have

$$\|v^{i,j}\|_\infty \leqslant d\left[\frac{\det\bar{A}}{\det\underline{A}}\,\frac{\lambda_n^{-1}(\bar{A})\cdots\lambda_{n-j+1}^{-1}(\bar{A})}{\lambda_n^{-1}(\underline{A})\cdots\lambda_{n-j+2}^{-1}(\underline{A})}\right]^i$$

$$= d\left[\delta_{n-j+1}(\underline{A},\bar{A})\right]^{-i}, \qquad i=0,1,2,\dots .$$

Hence the linearly independent vectors $u_-^{i,j}=Dv^{-i,n-j+1}$, $j=1,2,\dots,k$, satisfy the "forward" equation

$$x^{i+1}=A_i x^i, \qquad i\in Z,$$

and decay exponentially fast as $i\to-\infty$:

$$\|u_-^{i,j}\|\leqslant c_-\rho_-^{-i}, \qquad i=0,-1,-2,\dots, \quad j=1,\dots,k, \quad \rho_-<1.$$

Thus we have "decomposed" the solutions of the homogeneous equation into two subspaces of dimension k and $n-k$ which decay at $-\infty$ and $+\infty$, respectively. We claim that the vectors $u_-^{0,1},\dots,u_-^{0,k},u_+^{0,k+1},\dots,u_+^{0,n}$ are linearly independent. We argue this fact by contradiction. Suppose there exists a nonzero vector $\alpha=(\alpha_1,\dots,\alpha_n)$ such that

$$v=\sum_{j=1}^{k}\alpha_j u_-^{0,j}-\sum_{j=k+1}^{n}\alpha_j u_+^{0,j}=0, \tag{3.19}$$

then the sequence

$$x^i=\begin{cases}\displaystyle\sum_{j=k+1}^{n}\alpha_j u_+^{i,j}, & i\geqslant 1,\\ \displaystyle\sum_{j=1}^{k}\alpha_j u_-^{i,j}, & i\leqslant 0,\end{cases} \quad ,i\in Z, \tag{3.20}$$

is a bounded solution to the homogeneous equation, $x^{i+1}=A_i x^i$, $i\in Z$. Hence $x^0=x^1=0$, and we conclude $\alpha=0$, a contradiction.

Now, to construct the sequence for the lemma we note that any sequence expressible as (3.20) for some vector $\alpha=(\alpha_1,\dots,\alpha_n)$ satisfies $x^{i+1}-$

$A_i x^i = 0$, $i \in Z - \{0\}$. Thus we choose constants $\beta_1, \dots, \beta_n$ such that

$$\sum_{j=1}^{k} \beta_j u_-^{0,j} - \sum_{j=k+1}^{n} \beta_j u_+^{0,j} = A_0^{-1} b.$$

Now we claim that there exists a constant d, independent of b, such that

$$|\beta_j| \leqslant d \|b\|_\infty, \qquad j = 1, 2, \dots, n.$$

This will follow if we show that there is a positive constant depending only on $\underline{A}$ and $\overline{A}$ such that

$$\|u_-^{0,1} \wedge \cdots \wedge u_-^{0,k} \wedge u_+^{0,k+1} \wedge \cdots \wedge u_+^{0,n}\|_\infty \geqslant w_1 > 0.$$

The proof of this fact proceeds as in the proof of Lemma 3.3. If such an $w_1 > 0$ did not exist, we could find an operator $\langle A_i \rangle$ such that $\|u_-^{0,1} \wedge \cdots \wedge u_+^{0,n}\| = 0$, i.e., $u_-^{0,1}, \dots, u_+^{0,n}$ are linearly dependent. We have already shown this is impossible. Thus the sequence

$$x^i = \begin{cases} \displaystyle\sum_{j=k+1}^{n} \beta_j u_+^{i,j}, & i \geqslant 1, \\ \displaystyle\sum_{j=1}^{k} \beta_j u_-^{i,j}, & i \leqslant 0, \end{cases} \quad , i \in Z$$

satisfies the demands of the lemma.

The proof of part (b) proceeds by the same techniques, and we omit the details. ∎

We can now prove our main theorem.

Proof of Theorem 3.2. Fix $l \in Z$, and apply Lemma 3.4 to the difference equation corresponding to A_{l+i} and b^l. Thus there exist constants (independent of l) $c > 0$, $\rho < 1$ and $x^{l,i}$ such that

$$x^{l,1} - A_l x^{l,0} = b^l,$$

$$A_{l+i} x^{l,i} - x^{l,i+1} = 0, \qquad i \in Z - \{0\},$$

$$\|x^{l,i}\|_\infty \leqslant c \rho^{|i|} \|b^l\|_\infty, \qquad i, l \in Z.$$

Let $v^{l,i}=x^{l,i-l}$; then

$$v^{l,i+1}-A_i v^{l,i}=b^l\delta_{il}, \qquad i\in Z,$$

and

$$\|v^{l,i}\|_\infty \leqslant c\rho^{|i-l|}\|b^l\|_\infty.$$

Hence $x^i=\sum_{l=-\infty}^{\infty}v^{l,i}$ satisfies (2.1), and $\|x^i\|_\infty \leqslant 2Mc/(1-\rho)$, $i\in Z$, where $M=\sup_l\|b^l\|_\infty$. With this, the proof of Theorem 3.2 is completed. ∎

Motivated by our discussion of spline interpolation in Sec. 1, we consider the following application of Theorem 3.2.

Let $D(m)=\operatorname{diag}\{m,m^2,\dots,m^n\}$ and A an $n\times n$ oscillation matrix such that

$$DA^{-1}D=A. \tag{3.21}$$

Let $\{m_i : i\in Z\}$ be an infinite sequence in R^+, and define

$$A_i=AD(m_i).$$

We denote the eigenvalues of $AD(m)$ by $\lambda_1(m)>\lambda_2(m)>\cdots>\lambda_n(m)$.

THEOREM 3.6. *Let A be a $2k\times 2k$ oscillation matrix such that $DA^{-1}D=A$ and the eigenvalues of $AD(m)$ are strictly increasing functions of $m\in R^+$. Then $R(m)=m^{k(2k+1)}/\lambda_k(m)$ is a strictly increasing function of m which maps $[0,\infty)$ onto $[0,\infty)$. Let m^* be the unique zero of the equation*

$$R(m^*)=1;$$

then the difference equation

$$\begin{aligned} x^{i+1}&=A_i x^i+b^i, \qquad i\in Z,\\ A_i&=AD(m_i), \end{aligned} \tag{3.22}$$

is solvable if $m^{-1}\leqslant m_i\leqslant m$, $i\in Z$, where $m<m^*$.

The proof of this theorem follows by arguments analogous to those used in the proof of Theorem 1.1.

Finally, we remark that difference equations of the form (3.22) arise in various spline interpolation problems. The basic feature of this class of problems is that they are obtainable from a "fundamental" polynomial interpolation process on $[0,1]$ [in the example (0.1) we studied here, this is the procedure of the interpolating data $f(0),\dots,f^{(n-1)}(0),f(1)$ by a polynomial of degree $\leqslant n$] which is shifted and scaled onto an infinite partition Δ. It may be shown for a class of such problems that the matrix A is an oscillation matrix (see [6]). However, it is not known whether or not for this class the eigenvalues of $AD(m)$ are strictly increasing in m. For the special case $A=T$, *ad hoc* methods were used in [5] to prove the monotonicity of the eigenvalues. There are, however, some general observations to be made about the eigenvalues of $AD(m)$. We record them here.

THEOREM 3.7. *Let A be an oscillation satisfying* (3.21). *Then*:

(a) $\lambda_i(m^{-1})=\lambda_{n-i+1}^{-1}(m)$, $i=1,2,\dots n$.

(b) $\lambda_1(m)\cdots\lambda_n(m)=m^{n(n+1)/2}$.

(c) $\lambda_1(m)\cdots\lambda_i(m)/m\cdots m^i$ *is a strictly increasing function of* $m\in R^+$ *for any* i, $1\leqslant i<n$, *and* $\lambda_n(m)\cdots\lambda_{n-i+1}(m)/m\cdots m^i$ *is strictly increasing in* m *for any* i, $1\leqslant i\leqslant n$.

(d) *For any* i, $1\leqslant i\leqslant n$, *the function* $\lambda_i(m)m^{(i-1)(n-i)-1}$ *is increasing in* m.

Proof. Since $A^{-1}=DAD$, we have

$$\begin{aligned}[AD(m)]^{-1}&=D(m^{-1})A^{-1}\\&=D(m^{-1})DAD\\&=D(-m)^{-1}AD(m^{-1})D(-m).\end{aligned}$$

Thus $\sigma\big((AD(m))^{-1}\big)=\sigma(AD(m^{-1}))$, and the first assertion follows easily. Since (3.21) implies $\det A=1$, we have

$$\begin{aligned}\lambda_1(m)\cdots\lambda_n(m)&=\det[D(m)A]\\&=\det D(m)=m^{n(n+1)/2}.\end{aligned}$$

Our last claim, (d), follows from (c) as follows: fix i, $1\leqslant i\leqslant n$; then (c) implies that

$$\frac{\lambda_1(m)\cdots\lambda_i(m)}{m\cdots m^i}\,\frac{\lambda_n(m)\cdots\lambda_i(m)}{m\cdots m^{n-i+1}}=\frac{\lambda_i(m)m\cdots m^n}{m\cdots m^i m\cdots m^{n-i+1}}$$
$$=m^{(n-i)(i-1)-1}\lambda_i(m)$$

is strictly increasing.

The proof of (c) depends on the observation that the elements of the primitive matrix

$$\frac{\bigwedge_{i=1}^{r} AD(m)}{m\cdots m^r}$$

are strictly increasing functions of m. Using the max-min characterization of the largest eigenvalue of a nonnegative matrix (see [7]) we conclude that

$$\frac{\lambda_1(m)\cdots\lambda_i(m)}{m\cdots m^i},\qquad 1\leqslant i\leqslant n,$$

is a strictly increasing function of m. Similarly, the elements of the matrix

$$m\cdots m^i\bigwedge_{j=1}^{i} D(m^{-1})A$$

are decreasing functions of m. Thus $m\cdots m^i\lambda_n^{-1}(m)\cdots\lambda_{n-i+1}^{-1}(m)$ is a strictly decreasing function. Consequently

$$\frac{\lambda_n(m)\cdots\lambda_{n-i+1}(m)}{m\cdots m^i}$$

is increasing, and the proof of the theorem is complete. ∎

As a consequence of Theorem 3.7 we conclude that for $n=2$ and 3 the eigenvalues of $AD(m)$ are strictly increasing functions of m.

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